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module - 02

Conformal transformations and Complex integration

Bilinear Transformation (BLT)

The transformation $w = \frac{az+b}{cz+d}$ where a, b, c, d are real constants ∋; $ad - bc \neq 0$ is called a bilinear transformation.

working Procedure:

⇒ ① Given w_1, w_2, w_3 corresponding to z_1, z_2, z_3 we assume the billinear transformation in the form $w = \frac{az+b}{cz+d}$

② we substitute the given set of points to obtain a set of three equations in four unknowns a, b, c, d.

③ we deduce a pair of equation in any three unknowns and solve by the rule of cross x^n to obtain a proportionate set of values for three unknowns.

(4) The values are used to find the fourth unknown.

(5) All these four values when substituted in the assumed form of ω, will give us the required bilinear transformation.

Problems

(1) Find the bilinear transformation which map the points $z = 1, i, -1$ into $\omega = i, 0, -i$.

Sol^n: Let $\omega = \dfrac{az+b}{cz+d}$ —(1) be required bilinear transformation

when $z = 1$, $\omega = i$

eq^n (1) becomes

$$i = \frac{a+b}{c+d}$$

$$a+b = ci + di$$

$$a+b-ci-di = 0 \quad \text{—(2)}$$

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when $z = i,\ \omega = 0$

eqn ① becomes

$$0 = \frac{ai+b}{ci+d}$$

$$ai+b = 0 \quad — ③$$

when $z = -1,\ \omega = -i$

eqn ① becomes

$$-i = \frac{-a+b}{-c+d}$$

$$ci - di = -a+b$$

$$-a+b-ci+di = 0 \quad — ④$$

Solve ② & ④

$$\cancel{a}+b-ci-\cancel{di} = 0$$

$$-\cancel{a}+b-ci+\cancel{di} = 0$$

$$2b-2ci = 0$$

$$2(b-ic) = 0$$

$$b-ic = 0 \quad — ⑤$$

by cross x^n method

Solve ③ & ⑤

$$ai+b+0(c) = 0$$

$$a(0)+b-iC = 0$$

$$\frac{a}{\begin{vmatrix} 1 & 0 \\ 1 & -i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} i & 0 \\ 0 & -i \end{vmatrix}} = \frac{c}{\begin{vmatrix} i & 1 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{a}{-i} = \frac{-b}{1} = \frac{c}{i} = K \text{ (say)}$$

$a = -iK \quad -b = K \quad c = iK$

$b = -K$

put a, b, c in (2)

$$-iK - K - i^2K - di = 0$$

$$-iK - \cancel{K} + \cancel{K} - di = 0$$

$$-d\cancel{i} = \cancel{i}K$$

$$d = -K$$

put a, b, c, d in ω

$$\omega = \frac{-iKz - K}{iKz - K} = \frac{\cancel{-K}(+iz+1)}{\cancel{-K}(-iz+1)}$$

$$\omega = \frac{1+iz}{1-iz}$$

(2) Find the B.L.T which map the points $z = 1, i, -1$ into $\omega = 2, i, -2$ also find the invariant points of the transformation.

Solⁿ:
$$\omega = \frac{az+b}{cz+d} \quad \text{—(1)}$$

when $z=1,\ \omega=2$

Now ① becomes

$$2=\frac{a+b}{c+d}$$

$$a+b=2c+2d$$

$$a+b-2c-2d=0 \quad -②$$

when $z=i,\ \omega=i$

① becomes

$$i=\frac{ai+b}{ci+d}$$

$$ci^2+di=ai+b$$

$$-c+di=ai+b$$

$$ai+b+c-di=0 \quad -③$$

when $z=-1,\ \omega=-2$

① becomes

$$-2=\frac{-a+b}{-c+d}$$

$$2c-2d=-a+b$$

$$-a+b-2c+2d=0 \quad -④$$

~~Solve ② & ③~~

$$\cancel{a+b-2c-2d=0}\times i$$

$$\cancel{ai+b+c-di=0}$$

$$\cancel{ai+bi-2ci-2di=0}$$

$$\cancel{\underset{(-)}{}ai\ \underset{(-)}{+}b\ \underset{(-)}{+}c\ \underset{(+)}{-}di=0}$$

$$b(i-1)-c(2i$$

Solve ② & ④

$\cancel{a}+b-2c-2\cancel{d}=0$
$-\cancel{a}+b-2c+\cancel{2}d=0$

$2b-4c=0$
$2(b-2c)=0$
$b-2c=0$ — ⑤

Solve ③ & ④
$ai+b+c-di=0$
$-a+b-2c+2d=0 \times i$

$\cancel{ai}+b+c-di=0$
$-\cancel{ai}+bi-2ci+2di=0$

$b(1+i)+c(1-2i)+id=0$ — ⑥

Solve ⑤ & ⑥ by cross x^n method
$b-2c+0d=0$
$b(1+i)+c(1-2i)+id=0$

$$\frac{b}{\begin{vmatrix}-2 & 0\\1-2i & i\end{vmatrix}}=\frac{-c}{\begin{vmatrix}1 & 0\\1+i & i\end{vmatrix}}=\frac{d}{\begin{vmatrix}1 & -2\\1+i & 1-2i\end{vmatrix}}$$

$$\frac{b}{-2i-0}=\frac{-c}{i-0}=\frac{d}{1-\cancel{2i}+2+\cancel{2i}}=K$$

$$\frac{b}{-2i}=\frac{-c}{i}=\frac{d}{3}=K$$

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$b = -2iK$, $-c = iK$, $d = 3K$

$c = -iK$

put b, c, d in (2)

$a - \cancel{2iK} + \cancel{2iK} - 6K = 0$

$a - 6K = 0$

$a = 6K$ //

put all values in ω

$$\omega = \frac{6KZ - 2iK}{-iKZ + 3K}$$

$$= \frac{\cancel{K}(6Z - 2i)}{\cancel{K}(-iZ + 3)}$$

$$\omega = \frac{6Z - 2i}{-iZ + 3} //$$

To find the invariant points of this transformation are obtained by taking $\omega = Z$.

$$Z = \frac{6Z - 2i}{-iZ + 3}$$

$$-iZ^2 + 3Z = 6Z - 2i$$

$$-iZ^2 + 3Z - 6Z + 2i = 0$$

$$-iZ^2 - 3Z + 2i = 0$$

$$iZ^2 + 3Z - 2i = 0$$

$$z = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$= \frac{-3 \pm \sqrt{9-4 \times i \times -2i}}{2i}$$

$$= \frac{-3 \pm \sqrt{9+8i^2}}{2i}$$

$$= \frac{-3 \pm \sqrt{9-8}}{2i} \qquad (i^2=-1)$$

$$= \frac{-3 \pm \sqrt{1}}{2i}$$

$$= \frac{-3 \pm 1}{2i}$$

$$z = \frac{-3+1}{2i}, \qquad z = \frac{-3-1}{2i}$$

$$z = \frac{-\cancel{2}}{\cancel{2}i} \qquad z = \frac{-4}{2i}$$

$$z = \frac{-1}{i} \qquad z = \frac{-2}{i}$$

$$z = -(-i) \qquad z = -2\left(\frac{1}{i}\right)$$

$$z = i \qquad z = -2(-i)$$

$$z = 2i$$

$$\boxed{\begin{aligned} \frac{1}{i} &= \frac{1}{i} \times \frac{-i}{-i} \\ &= \frac{-i}{-i^2} \\ &= \frac{-i}{-(-1)} = -i \\ \therefore \frac{1}{i} &= -i \end{aligned}}$$

$\therefore$ Invariant points are
$i, 2i$

(3) Find the B.L.T which maps $z_1=-1$, $z_2=0$, $z_3=1$ into $w_1=0$, $w_2=i$, $w_3=3i$

Soln:
$$w = \frac{az+b}{cz+d} \quad \text{--- (1)}$$

when $w=0$ $z=-1$

(1) becomes
$$0 = \frac{-a+b}{-c+d}$$
$$-a+b=0 \quad \text{--- (2)}$$

when $w=i$, $z=0$

$$i = \frac{0+b}{0+d} \Rightarrow b = id \Rightarrow b-id=0 \text{ --- (3)}$$

when $w=3i$, $z_3=1$

(1) becomes
$$3i = \frac{a+b}{c+d}$$
$$3ci+3di = a+b$$
$$a+b-3ci-3di=0 \quad \text{--- (4)}$$

Solve (2) & (3)
$$-a+\cancel{b}=0$$
$$\underset{(-)}{\cancel{b}} \underset{(+)}{-} di = 0$$
$$-a+di=0 \quad \text{--- (5)}$$

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Solve ③ & ⑤ by cross x^n method

$0(a) + b - id = 0$

$-a + 0(b) + id = 0$

$$\frac{a}{\begin{vmatrix} 1 & -i \\ 0 & i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 0 & -i \\ -1 & i \end{vmatrix}} = \frac{c}{\begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}}$$

$$\frac{a}{i} = \frac{-b}{-i} = \frac{d}{1} = K$$

$a = i \quad -b = -iK \quad d = K$

$a = i, \quad b = iK \quad d = K$

put in ④

$iK + iK - 3ci - 3Ki = 0$

$-3ci - Ki = 0$

$-i(3c + K) = 0$

$3c + K = 0$

$3c = -K$

$c = -K/3$ //

put all values in ①

$$\omega = \frac{i zK + iK}{-K/3\, z + K}$$

$$\omega = \frac{i\cancel{K}(z+1)}{\cancel{K}(-z/3+1)}$$

$$\omega = \frac{i(z+1)}{(-z/3+1)}$$

(or) $$\omega = \frac{i(z+1)}{\left(\frac{-z+3}{3}\right)}$$

$$\omega = \frac{3i(z+1)}{-z+3}$$

(4) Find the B.L.T which maps $z=\infty, i, 0$ into $\omega = -1, -i, 1$. Also find the fixed points of transformations. (Invariant points)

Sol^n:

$$\omega = \frac{az+b}{cz+d}$$

$$\omega = \frac{z(a+b/z)}{z(c+d/z)}$$

$$\omega = \frac{a+b/z}{c+d/z} \quad \text{—} \circledast$$

when $z=\infty$, $\omega = -1$

(*) becomes $-1 = \frac{a+0}{c+0}$

$-1 = \frac{a}{c} \Rightarrow a = -c \Rightarrow a+c = 0$ — (1)

when $z = i$, $w = -i$

(*) becomes $-i = \frac{a + b/i}{c + d/i}$

$$-i = \frac{\frac{ai+b}{\cancel{i}}}{\frac{ci+d}{\cancel{i}}}$$

$$-i = \frac{ai+b}{ci+d}$$

$$-i(ci) - di = ai + b$$

$$-ci^2 - di = ai + b$$

$$-c(-1) - di = ai + b$$

$$c - di = ai + b$$

$$ai + b - c + di = 0 \quad \text{— (2)}$$

when $z = 0$, $w = 1$

(*) becomes $1 = \frac{b}{d}$

$$b = d \Rightarrow b - d = 0 \quad - (3)$$

Solve (1) & (2)

$$a + 0(b) + \cancel{c} + 0d = 0$$
$$ai + b - \cancel{c} + di = 0$$
$$a(1+i) + b + di = 0 \quad - (4)$$

by cross x^n solve (3) & (4)

$$a(0) + b - d = 0$$
$$a(1+i) + b + di = 0$$

$$\frac{a}{\begin{vmatrix} 1 & -1 \\ 1 & i \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 0 & -1 \\ 1+i & i \end{vmatrix}} = \frac{d}{\begin{vmatrix} 0 & 1 \\ 1+i & 1 \end{vmatrix}}$$

$$\frac{a}{i+1} = \frac{-b}{0+1+i} = \frac{d}{-(1+i)} = K$$

$a = K(1+i)$ $\qquad -b = K(+1+i),\ d = -K(1+i)$

$a = K(1+i)$ $\qquad b = -K(1+i)$

$\qquad b = -K(1+i)$

put a, b, d valuey in (2)

$$K(1+i)i - K(1+i) - c - K(1+i)i = 0$$

$$K(i+i^2) + K(1+i) - c + K(i+i^2) = 0$$

$i^2 = -1$

$$k(i-1) - k(1+i) - c - k(i-1) = 0$$

$$\cancel{ik} - \cancel{k} - k - \cancel{ik} - c - ik + \cancel{k} = 0$$

$$-k - c - ik = 0$$

$$-c - ik - k = 0$$

$$-c = k + ik$$

$$c = -k - ik$$

$$c = -k(1+i)$$

put a, b, c, d in w

$$w = \frac{az+b}{cz+d}$$

$$= \frac{k(1+i)z + (-k)(1+i)}{-k(1+i)z - k(1+i)}$$

$$= \frac{\cancel{k(1+i)}(z-1)}{\cancel{k(1+i)}(-z-1)}$$

$$= \frac{-(1-z)}{-(z+1)}$$

$$w = \frac{1-z}{1+z}$$

To find invariant points take $\omega = z$ in (*)

$$z = \frac{1-z}{1+z}$$

$$z + z^2 = 1 - z$$

$$z^2 + z + z - 1 = 0$$

$$z^2 + 2z - 1 = 0$$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{4+4}}{2}$$

$$= \frac{-2 \pm \sqrt{8}}{2}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2}$$

$$z = -1 \pm \sqrt{2}$$

$\therefore$ invariant points are $-1+\sqrt{2}$, $-1-\sqrt{2}$ //

5) Do yourself

Find the B.L.T which maps the points $z = 1, i, -1$ into $\omega = 0, 1, \infty$

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$$w = \frac{az+b}{cz+d}$$

$z=1,\ w=0;$

(*) becomes

$$0 = \frac{a+b}{c+d}$$

$a+b=0$ — (1)

$z=i,\ w=1;\ 1 = \dfrac{ai+b}{ci+d}$

$ai+b-ci-d=0$ — (2)

$z=-1,\ w=\infty$

consider $\dfrac{1}{w} = \dfrac{cz+d}{az+b}$, $\left(\dfrac{1}{\infty}=0\right)$

$\left(\because \dfrac{1}{w} = \dfrac{1}{\infty}=0\right)$

$$0 = \frac{-c+d}{-a+b}$$

$-c+d=0$ — (3)

solve (2) & (3)

$$\begin{array}{r} ai+b-ci-\cancel{d}=0 \\ -c+\cancel{d}=0 \\ \hline ai+b-c(i+1)=0 \end{array}$$ — (4)

Solving (1) & (4) by cross x^n method

$a+b+0c=0$

$ai+b-c(i+1)=0$

$$\frac{a}{\begin{vmatrix} 1 & 0 \\ 1 & -(i+1) \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 1 & 0 \\ a & -(i+1) \end{vmatrix}} = \frac{c}{\begin{vmatrix} 1 & 1 \\ i & 1 \end{vmatrix}}$$

$$\frac{a}{-(i+1)} = \frac{-b}{-(i+1)} = \frac{c}{1-i} = K$$

$$\frac{a}{-(i+1)} = K, \quad \frac{b}{i+1} = K, \quad \frac{c}{1-i} = K$$

$a = -K(i+1) \quad b = K(i+1) \quad c = K(1-i)$

put a, b, c in ②

$-K(i+1)i + K(i+1) - K(1-i)i - d = 0$

$-Ki^2 - Ki + \cancel{Ki} + K + Ki^2 - \cancel{Ki} - d = 0$

$-K(-1) - iK + K + K(-1) - d = 0$

$+\cancel{K} - iK + K - \cancel{K} - d = 0$

$K(1-i) - d = 0$

$\underline{d = K(1-i)}$

put all values in ω

$$\omega = \frac{-\cancel{K}\{(1+i)z + (1+i)\}}{\cancel{K}\{(1-i)z + (1-i)\}}$$

$$\omega = \frac{(1+i)(1-z)}{(1-i)(1+z)}$$

$\times^{ly}$ & $\div$ by $1+i$

$$\omega = \frac{(1+i)(1-z)(1+i)}{(1-i)(1+z)(1+i)}$$

$$= \frac{(1+i)^2 (1-z)}{(1-i^2)(1+z)}$$

$$= \frac{(1+i^2+2i)(1-z)}{(1+1)(1+z)}$$

$$= \frac{(\cancel{1}-\cancel{1}+2i)(1-z)}{(2)(1+z)}$$

$$\omega = i\left[\frac{1-z}{1+z}\right] //$$

(6) Find the B.L.T which map the points $z=0, 1, \infty$ into the points $w=-5, -1, 3$ respectively. what are the invariant points?

Soln: $\omega = \frac{az+b}{cz+d}$ — (*)

$z=0, \ \omega=-5$

(*) becomes $-5 = \frac{b}{d}$

$b = -5d$ — (**)

$b+5d=0$ — (1)

$z=1, \ \omega=-1$

(*) becomes $-1 = \frac{a+b}{c+d}$

$-c-d = a+b$

$a+b+c+d=0 \quad - (2)$

$z=\infty, \; w=3$

$$w = \frac{\cancel{z}(a+b/z)}{\cancel{z}(c+d/z)}$$

$$w = \frac{(a+b/\infty)}{(c+d/\infty)}$$

$$3 = \frac{a+0}{c+0}$$

$$3 = \frac{a}{c} \Rightarrow a = 3c \quad - (***)$$

$-3c+a=0 \quad - (3)$

Solve (2) & (3)

$$\begin{array}{r} a+b+c+d=0 \\ (-)\; a \pm 0b \mp 3c \pm 0d = 0 \\ \hline b+4c+d=0 \quad - (4) \end{array}$$

Solve (1) & (4) by cross x^n method

$$\begin{array}{r} b+0c+5d=0 \\ b+4c+d=0 \\ \hline \end{array}$$

$$\frac{b}{\begin{vmatrix} 0 & 5 \\ 4 & 1 \end{vmatrix}} = \frac{-c}{\begin{vmatrix} 1 & 5 \\ 1 & 1 \end{vmatrix}} = \frac{d}{\begin{vmatrix} 1 & 0 \\ 1 & 4 \end{vmatrix}}$$

$$\frac{b}{-20} = \frac{-c}{-4} = \frac{d}{4} = K$$

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$\frac{b}{-20} = K, \quad \frac{c}{4} = K, \quad \frac{d}{4} = K$

$b = -20K, \quad c = 4K, \quad d = 4K$

put b, c, d in (2)

$a - 20K + 4K + 4K = 0$

$a - 12K = 0$

$a = 12K$ //

$\omega = \frac{\cancel{K}(12z - 20)}{\cancel{K}(4z + 4)}$

$= \frac{\cancel{4}(3z - 5)}{\cancel{4}(z + 1)}$

$\omega = \frac{3z - 5}{z + 1}$ //

or use $(**)$ & $(***)$ in (2)

(2) $\Rightarrow$ $3c - 5d + c + d = 0$

$4c - 4d = 0$

$4(c - d) = 0$

$c - d = 0$

$\underline{\underline{c = d}}$

put $c = d$ in (2)

$a + b + d + d = 0$

$a + b + 2d = 0$ — (5)

Scanned with CamScanner

Solve (1) & (5) by cross x^n method

$0(a) + b + 5d = 0$

$a + b + 2d = 0$

$$\frac{a}{\begin{vmatrix} 1 & 5 \\ 1 & 2 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 0 & 5 \\ 1 & 2 \end{vmatrix}} = \frac{d}{\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}}$$

$$\frac{a}{2-5} = \frac{-b}{-5} = \frac{d}{-1} = K$$

$$\frac{a}{-3} = K, \quad \frac{b}{5} = K, \quad \frac{d}{-1} = K$$

$a = -3K, \ b = 5K, \ d = -K$

(2) $\Rightarrow$ $-3K + 5K + c - K = 0$

$K + c = 0$

$c = -K$ //

$$\omega = \frac{\cancel{K}(-3z + 5)}{\cancel{K}(-z - 1)}$$

$$= \frac{+(3z - 5)}{+(z + 1)}$$

$$\omega = \frac{3z - 5}{z + 1} \ //$$

To find invariant points put $w = z$

$$z = \frac{3z-5}{z+1}$$

$$z^2 + z = 3z - 5$$

$$z^2 + z - 3z + 5 = 0$$

$$z^2 - 2z + 5 = 0$$

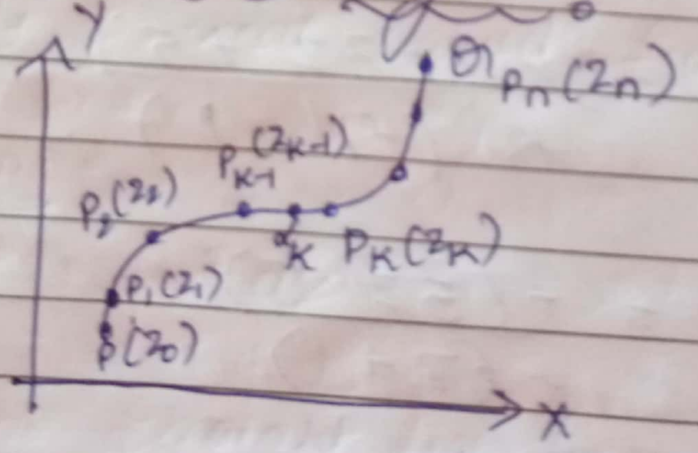
$$z = \frac{2 \pm \sqrt{4-20}}{2}$$

$$= \frac{2 \pm \sqrt{-16}}{2}$$

$$= \frac{2 \pm i4}{2}$$

$z = 1 \pm 2i$ are the invariant points.

Complex line integral :



Consider a Contineous function $f(z)$ of the Complex variable $z = x+iy$ defined at all points of a curve

C extending from P to Q. Divide the curve C into n parts by arbitarily taking points $P = P(z_0), P_1(z_1), P_2(z_2) \ldots P_K(z_K) \cdots P_n(z_n) = Q$ on the curve C. Let α_K be any point on the arc of the curve from P_{K-1} to P_K and let $\delta z_K = z_K - z_{K-1}$ where $K = 1, 2, 3 \ldots n$

Then

$$\lim_{n \to \infty} \sum_{K=1}^{n} f(\alpha_K)\,\delta z_K \text{ where } \max|\delta z_K| \to 0$$

as $n \to \infty$ is defined as complex line integral along the path C usually denoted by $\int_C f(z)\,dz$.

Properties of Complex integral :

① If $-C$ denotes the curve traversed from Q to P then $\int_{-C} f(z)\,dz = -\int_C f(z)\,dz$

② If C is split into a no. of parts $C_1, C_2, C_3 \ldots$ then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz + \int_{C_3} f(z)\,dz + \cdots$$

③ If λ_1 & λ_2 are constants then

$$\int_C [\lambda_1 f_1(z) \pm \lambda_2 f_2(z)]\, dz = \lambda_1 \int_C f_1(z)\, dz \pm \lambda_2 \int_{C_2} f_2(z)\, dz$$

Line integral of a Complex valued function

Let $f(z) = u(x,y) + iv(x,y)$ be a complex valued function defined over a region R and C be a curve in the region. Then

$$\int_C f(z)\, dz = \int_C (u+iv)(dx + i\, dy)$$

$$\int_C f(z)\, dz = \int_C (u\,dx - v\,dy) + i \int_C (v\,dx + u\,dy).$$

This shows that evaluation of a line integral of a complex valued function is nothing but the evaluation of line integrals of real valued functions.

Problems:

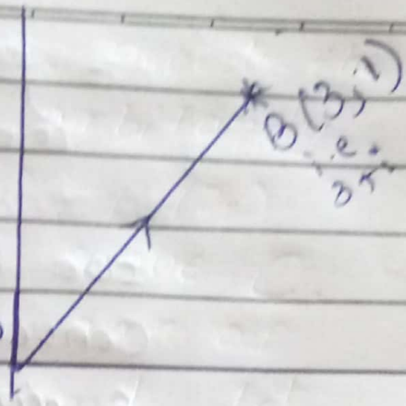
(1) Evaluate $\int_C z^2\, dz$

(a) along the straight line from $z=0$ to $3+i$

(b) along the curve madeup of two line segments one from $z=0$ to 3 & another from $z=3$ to $3+i$

z) dz

Soln: $\int_C z^2 dz = \int_{z=0}^{3+i} z^2 dz$



here z varies from 0 to 3+i
means (x, y) vary from (0,0) to (3,1).

The eqn of line joining the points (0,0) and (3,1) is given by

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

$$\frac{y-0}{x-0} = \frac{1-0}{3-0}$$

$$\frac{y}{x} = \frac{1}{3} \text{ or } y = \frac{x}{3}$$

$$z^2 = (x+iy)^2$$

$$= x^2 + i^2y^2 + 2xiy$$

$$z^2 = x^2 - y^2 + i2xy$$

$$dz = dx + idy$$

$$\int_C z^2 dz = \int_{(0,0)}^{(3,1)} \{(x^2-y^2) + i2xy\}(dx+idy)$$

$$= \int_{(0,0)}^{(3,1)} \{(x^2-y^2)dx + i(x^2-y^2)dy + i2xy\,dx + i^2 2xy\,dy\}$$

$$= \int_{(0,0)}^{(3,1)} (x^2-y^2)dx + i(x^2-y^2)dy + i2xy\,dx - 2xy\,dy$$

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$$= \int_{(0,0)}^{(3,1)} \{(x^2-y^2)dx - 2xy\,dy\} + i\int_{(0,0)}^{(3,1)} \{(x^2-y^2)dy + 2xy\,dx\}$$

we have $y = \frac{x}{3}$ or $x = 3y$ & we shall Convert thye integral into the variable y & integrate w.r. to y from 0 to 1 & also $dx = 3dy$

$$\int_c z^2 dz = \int_{y=0}^{1} \{(9y^2-y^2)3dy - 2(3y)y\,dy\} + i\int_{y=0}^{1} \{(9y^2-y^2)dy + 2(3y)y3dy\}$$

$$= \int_{y=0}^{1} (24y^2dy - 6y^2dy) + i\int_{y=0}^{1} 8y^2dy + 18y^2dy$$

$$= \int_{y=0}^{1} 18y^2dy + i\int_{y=0}^{1} 26y^2dy$$

$$= 18\left[\frac{y^3}{3}\right]_{y=0}^{1} + i26\left[\frac{y^3}{3}\right]_{y=0}^{1}$$

$$= 18[1/3 - 0] + i26[1/3 - 0]$$

$$= \cancel{18}^{6}/\cancel{3} + i\,26/3$$

$= \underline{6 + i\,26/3}$ along the given path

along

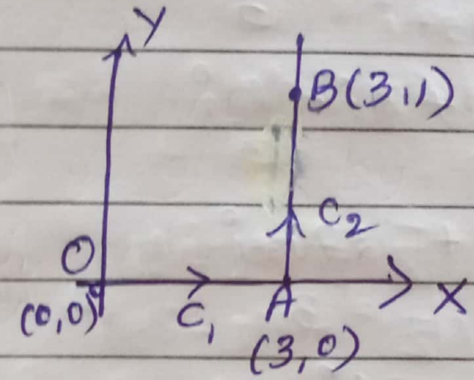
along

$(x+iy)$

$(3+iy)$

$2xy\,dx\}$

(b) Segments from $z=0$ to $z=3$ and then from $z=3$ to $3+i$ means (x,y) vary from $(0,0)$ to $(3,0)$ & then from $(3,0)$ to $(3,1)$



$$\int_C z^2 dz = \int_{C_1} z^2 dz + \int_{C_2} z^2 dz \quad \text{—— (1)}$$

along C_1 : $y=0 \Rightarrow dy=0$ & x vary from 0 to 3
$z^2 dz$ becomes $x^2 dx$

along C_2 : $x=3 \Rightarrow dx=0$ & y vary from 0 to 1
$z^2 dz$ becomes $(3+iy)^2 i\,dy$.

Now (1) becomes

$$\int_C z^2 dz = \int_{x=0}^{3} x^2 dx + i\int_{y=0}^{1} \underbrace{(3+iy)^2}_{(a+b)^2} dy$$

$$= \left[\frac{x^3}{3}\right]_{x=0}^{3} + i\int_{y=0}^{1} (9-y^2+6iy)\,dy$$

$(x+iy)^2$

$(3+iy)^2$

$$= 9 + i\left[9y - \frac{y^3}{3} + 3iy^2\right]_0^1$$

$$= 9 + i\left(9 - \frac{1}{3} + 3i - 0\right)$$

$$= 9 + 9i - \frac{1}{3}i - 3$$

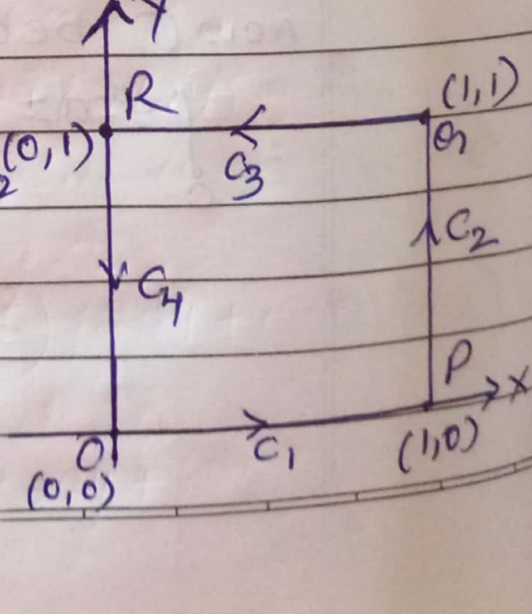
$$= 6 + 9i - \frac{1}{3}i$$

$$= 6 + \frac{27i - i}{3}$$

$$= 6 + \frac{26i}{3}$$

$\int_C z^2 dz = 6 + i\frac{26}{3}$ // along the given path

(3) Evaluate $\int_C |z|^2 dz$ where C is a square with following vertices (0,0) (1,0) (1,1) (0,1)



Sol^n:

$$\int_C |z|^2 dz = \int_{C_1} |z|^2 dz + \int_{C_2} |z|^2 dz + \int_{C_3} |z|^2 dz + \int_{C_4} |z|^2 dz$$

——— (1)

we have $|z|^2 dz = (\sqrt{x^2+y^2})^2 (dx+idy)$

$|z|^2 dz = (x^2+y^2) dx + idy$

along OP (C_1), $y=0 \Rightarrow dy=0$. $|z|^2 dz = x^2 dx$ when $0 \le x \le 1$

along PQ (C_2), $x=1 \Rightarrow dx=0$. $|z|^2 dz = (1+y^2) i dy$, $0 \le y \le 1$

along QR (C_3), $y=1 \Rightarrow dy=0$. $|z|^2 dz = (x^2+1) dx$, $1 \le x \le 0$

along RO (C_4), $x=0 \Rightarrow dx=0$. $|z|^2 dz = y^2 (idy)$ where $1 \le y \le 0$

use these results in ①

$$\int_C |z|^2 dz = \int_{x=0}^{1} x^2 dx + i\int_{y=0}^{1} (1+y^2) dy + \int_{x=1}^{0} (x^2+1) dx$$

$$+ i\int_{y=1}^{0} y^2 dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 + i\left[y + \frac{y^3}{3}\right]_0^1 + \left[\frac{x^3}{3} + x\right]_1^0 + i\left[\frac{y^3}{3}\right]_1^0$$

$$= \left(\frac{1}{3} - 0\right) + i\left(1 + \frac{1}{3} - 0\right) + \left(0 - \left(\frac{1}{3} + 1\right)\right) + i\left(0 - \frac{1}{3}\right)$$

$$= \frac{1}{3} + i\left(\frac{4}{3}\right) - \frac{4}{3} + i\left(-\frac{1}{3}\right)$$

$$= \frac{1}{3} + \frac{i4}{3} - \frac{4}{3} - \frac{i}{3}$$

$$= \frac{1 + 4i - 4 - i}{3} = \frac{-3 + 3i}{3} = -1 + i$$

$$\int_C |z|^2 dz = -1 + i \text{ along the given path}$$

③ Evaluate $\int_0^{2+i} (\bar{z})^2 \, dz$ along:

ⓐ the line $x = 2y$

ⓑ the real axis upto 2 & then vertically to $2+i$

Sol^{n}: let $I = \int_0^{2+i} (\bar{z})^2 dz$

we have $\bar{z} = x - iy$

$$(\bar{z})^2 = (x-iy)^2$$
$$= x^2 + (iy)^2 - 2xiy$$
$$= x^2 + i^2 y^2 - 2xiy$$
$$(\bar{z})^2 = x^2 - y^2 - i2xy$$
$$(\bar{z})^2 = (x^2 - y^2) - i(2xy)$$
$$dz = dx + i\,dy$$

ⓐ along $x = 2y$, $dx = 2dy$

$z = 0$ to $2+i \Rightarrow (x, y)$ vary from $(0,0)$ to $(2,1)$ where $0 \le y \le 1$

$$I = \int_{y=0}^{1} \left(\left((2y)^2 - y^2\right) - i\left(2(2y)\,y\right)\right)(2dy + idy)$$

$$= \int_{y=0}^{1} \left(4y^2 - y^2 - i4y^2\right)(2dy + idy)$$

$$= \int_{y=0}^{1} \left(3y^2 - i4y^2\right)(2dy + idy)$$

$$= \int_{y=0}^{1} (3-4i)y^2 \cdot dy\,(2+i)$$

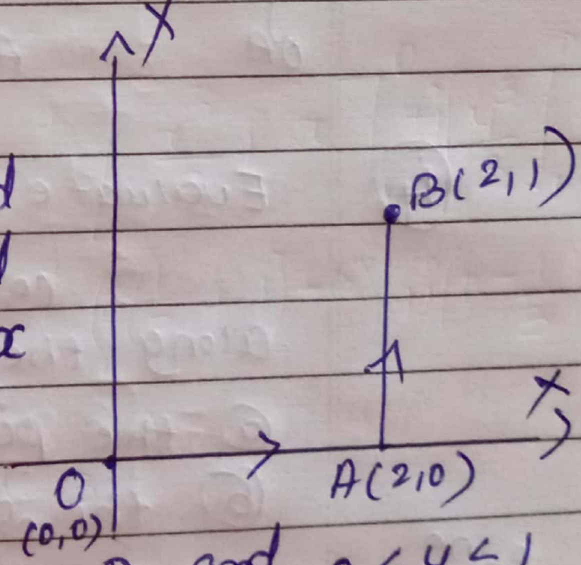
$$= (3-4i)(2+i)\int_{y=0}^{1} y^2 \cdot dy$$

$$= (6+3i-8i-4i^2)\left[y^3/3\right]_0^1$$

$$= (6-5i+4)\left[1/3-0\right]$$

$$I = (10-5i)\frac{1}{3} \text{ along the given path.}$$

(b) $$I = \int_{OA} (\bar{z})^2\, dz + \int_{AB} (\bar{z})^2\, dz \quad \text{--- (1)}$$



Along OA where $O=(0,0)$ and $A=(2,0)$; $y=0 \Rightarrow dy=0$ and $0 \le x \le 2$ and $(\bar{z})^2 dz = x^2 dx$

Along AB where $A=(2,0)$ and $B=(2,1)$; $x=2 \Rightarrow dx=0$ and $0 \le y \le 1$

$$(\bar{z})^2 dz = (2^2-y^2) - i(4y)$$

$$(\bar{z})^2 dz = (4-y^2) - i4y$$

Now ① becomy

$$\int_{OA} (\bar{z})^2 dz = \int_{x=0}^{2} x^2 dx = \frac{x^3}{3}\Big]_0^1 = \frac{8}{3}$$

$$\int_{AB} (\bar{z})^2 dz = i\int_{y=0}^{1} [(4-y^2) - 4iy] dy$$

$$\left[4y - \frac{y^3}{3}\right]_0^1 + 4\left[\frac{y^2}{2}\right]_0^1$$

$$= i\left[\left(4 - \frac{1}{3}\right) - 0\right] + 4\left[\frac{1}{2} - 0\right]$$

$$\int_{AB} (\bar{z})^2 dz = i\frac{11}{3} + 2$$

∴ ① becomy

$$\int_{OA} z^2 dz + \int_{AB} z^2 dz = 8 + \left(i\frac{11}{3} + 2\right)$$

④ Evaluate $\int_{(0,3)}^{(2,4)} (2y + x^2)dx + (3x - y)dy$

along the following paths

ⓐ the parabola $x = 2t$, $y = t^2 + 3$

ⓑ the straight line from $(0,3)$ to $(2,4)$

Solⁿ:

ⓐ x varies from 0 to 2 and hence

$$\left.\begin{array}{l} \text{If } x = 0,\ 2t = 0 \ \therefore t = 0 \\ \text{If } x = 2,\ 2t = 2 \ \therefore t = 1 \end{array}\right\} \Rightarrow t \text{ also varies from } 0 \text{ to } 1$$

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$$I = \int_{(0,3)}^{(2,4)} (2y + x^2)\,dx + (3x - y)\,dy$$

$$I = \int_{t=0}^{1} \left\{ 2(t^2+3) + 4t^2 \right\} 2\,dt + \left\{ 3(2t) - (t^2+3) \right\} 2t\,dt$$

$$= \int_0^1 (2t^2 + 6 + 4t^2)\,2\,dt + (6t - t^2 - 3)\,2t\,dt$$

$$= \int_0^1 (6t^2 + 6)\,2\,dt + (6t - t^2 - 3)\,2t\,dt$$

$$= \int_0^1 (12t^2 + 12 + 12t^2 - 2t^3 - 6t)\,dt$$

$$= \int_0^1 (24t^2 - 2t^3 - 6t + 12)\,dt$$

$$= \left[\frac{\cancel{24}^{8} \cdot t^3}{\cancel{3}} - \frac{2t^4}{4} - \frac{6t^2}{2} + 12t \right]_0^1$$

$$= 8[1-0] - \frac{1}{2}(1-0) - 3(1-0) + 12(1-0)$$

$$= 8 - \frac{1}{2} - 3 + 12$$

$$= 17 - \frac{1}{2}$$

$$= \frac{34-1}{2}$$

$I = 33/2$ // along the given path

(b) Equation of straight line joining (0, 3) & (2, 4) is given by

$$\frac{y-3}{x-0} = \frac{4-3}{2-0}$$

$$\frac{y-3}{x} = \frac{1}{2}$$

$$x = 2y-6$$

$$dx = 2dy$$

$$I = \int_{y=3}^{4} \{2y + \underbrace{(2y-6)^2}_{(a-b)^2}\} 2dy + \{3(2y-6)-y\}dy$$

$$= \int_3^4 \{2(2y+4y^2+36-2\times 2y\times 6) + (6y-18-y)\} dy$$

$$= \int_3^4 \{(4y+8y^2+72-48y) + (5y-18)\} dy$$

$$= \int_3^4 (8y^2-39y+54)\, dy$$

$$= \left[8\frac{y^3}{3} - 39\cdot\frac{y^2}{2} + 54y\right]_3^4$$

$$= \frac{8}{3}[4^3-3^3] - \frac{39}{2}[4^2-3^2] + 54[4-3]$$

$$= \frac{8}{3}[64-27] - \frac{39}{2}[16-9] + 54$$

$I = \frac{97}{6}$ along the given path

(5) Evaluate $\int_{1-i}^{2+i} (2x+iy+1)\,dz$ along the following paths:

(a) $x = t+1,\ y = 2t^2-1$

(b) straight line joining $(1-i)$ and $(2+i)$

Sol: (a) $x = t+1,\ y = 2t^2-1$

$dx = dt,\ dy = 4t\,dt$

x vary from 1 to 2

If $x = 1,\ t = 0$

$x = 2,\ t = 1$

$dz = dx + i\,dy,\ dz = dt + i4t\,dt$

$I = \int_{t=0}^{1} \{2(t+1) + i(2t^2-1) + 1\}\{dt + i4t\,dt\}$

$= \int_0^1 (2t + 2 + 1 + i2t^2 - i)\,dt\,(1+i4t)$

$= \int_0^1 (2t + 2it^2 - i + 3)(1+i4t)\,dt$

$= \int_0^1 \{2t + i8t^2 + 2it^2 + i^2 8t^3 - i - i^2 4t + 3 + i12t\}\,dt$

$= \int_0^1 \{2t + i10t^2 - 8t^3 - i - (-1)4t + 3 + i12t\}\,dt$

$= \int_0^1 \{2t + 4t + i10t^2 - 8t^3 - i + 3 + i12t\}\,dt$

$= \int_0^1 \{6t + i10t^2 - 8t^3 - i + 3 + i12t\}\,dt$

$$\left[\frac{6t^2}{2} + i\,10\frac{t^3}{3} - 8\frac{t^4}{4} - it + 3t + i\,12\frac{t^2}{2}\right]_0^1$$

$$\left[3t^2 + i\frac{10}{3}t^3 - 2t^4 - it + 3t + i\,6t^2\right]_0^1$$

$$\left[\underline{3} + i\frac{10}{3} - \underline{2} - i + \underline{3} + i6\right]$$

$$= 4 + i\left(\frac{10}{3} - 1 + 6\right)$$

$$= 4 + i\left(\frac{10-3+18}{3}\right)$$

$$I = 4 + i\frac{25}{3} \text{ along the given path.}$$

Equation of the straight line joining (1, -1) and (2, 1) is given by

$$\frac{y+1}{x-1} = \frac{1-(-1)}{2-1}$$

$$\frac{y+1}{x-1} = \frac{2}{1}$$

$$y+1 = 2x-2$$

$$y = 2x-2-1$$

$$y = 2x-3$$

$$dy = 2dx$$

$$I = \int_{x=1}^{2}\{2x + i(2x-3)+1\}\{dx + i2dx\}$$

$$= \int_{x=1}^{2}\{2(1+i)x + (1-3i)\}(1+2i)dx$$

$$= \int_{x=1}^{2} \{2(1+i)(1+2i)x + (1-3i)(1+2i)\}\,dx$$

$$I = \int_{x=1}^{2} \{2(-1+3i)x + (1-3i)(1+2i)\}\,dx$$

$$= (1-3i)\int_{x=1}^{2} \{-2x + (1+2i)\}\,dx$$

$$= (1-3i)\left\{-\cancel{2}\left(\frac{x^2}{\cancel{2}}\right)_1^2 + (x+2ix)_1^2\right\}$$

$$= (1-3i)\{-(4-1) + (2+4i-1-2i)\}$$

$$= (1-3i)\{-3+1+2i\}$$

$$= (1-3i)(-2+2i)$$

$$= -2+2i+6i-6i^2$$

$$= -2+8i-6(-1)$$

$$= -2+8i+6$$

$I = \underline{\underline{4+8i}}$ along the given path dd

Cauchy's theorem:

statement: If $f(z)$ is analytic at all points inside and on a simple closed curve C then $\int_C f(z)\,dz = 0$

proof: Let $f(z) = u + iv$

$$\int_C f(z)\,dz = \int_C (u+iv)(dx+idy)$$

$$\int_C f(z)\,dz = \int_C (\underset{M}{u}\,dx - \underset{N}{v}\,dy) + i\int_C (\underset{M}{v}\,dx + \underset{N}{u}\,dy) \quad \text{— (1)}$$

$M = u$

$\frac{\partial M}{\partial y} = \frac{\partial u}{\partial y}$

$N = -v$

$\frac{\partial N}{\partial x} = -\frac{\partial v}{\partial x}$

$M = v$

$\frac{\partial M}{\partial y} = \frac{\partial v}{\partial y}$

$N = u$

$\frac{\partial N}{\partial x} = \frac{\partial u}{\partial x}$

we have Green's theorem in a plane stating that if $M(x,y)$ & $N(x,y)$ are two real valued functions having continuous first order partial derivatives in a region R bounded by curve C then

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

Applying this theorem to the two lines integrals in the RHS of (1) we obtain

$$\int_C f(z)\,dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Since $f(z)$ is analytic we have Cauchy-Riemann equations $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ & hence we have

$$\int_C f(z)\,dz = \iint_R \left(\frac{\partial u}{\partial y} - \frac{\partial u}{\partial y}\right) dx\,dy + i \iint_R \left(\frac{\partial v}{\partial y} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$$\int_C f(z)\,dz = 0$$

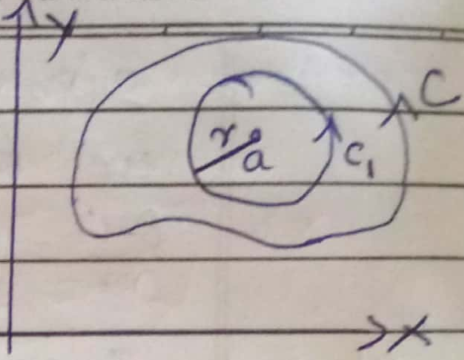
This proves Cauchy's theorem.

Cauchy's integral formula

Statement: If $f(z)$ is analytic inside and on a simple closed curve C & if 'a' is any point within C then

$$f(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z-a}\,dz$$

proof: Since 'a' is a point within C, we shall enclose it by a circle C_1 with $z=a$ as centre and r as radius r; C_1 lies entirely within C.



The function $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the annular region between C & C_1

Now as a consequence of Cauchy's theorem

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{C_1} \frac{f(z)}{z-a}\,dz \quad \text{— (1)}$$

The equation of C_1 (O'le with centre 'a' & radius r) can be written in the form $|z-a| = r$. This is equivalent to $z-a = re^{i\theta}$

$$z = a + re^{i\theta} \quad 0 \le \theta \le 2\pi.$$

$$dz = ire^{i\theta}\,d\theta$$

use these results in the RHS of (1)

$$\int_C \frac{f(z)}{z-a}\,dz = \int_{\theta=0}^{2\pi} \frac{f(a+re^{i\theta})\, i\cancel{re^{i\theta}}\,d\theta}{\cancel{re^{i\theta}}}$$

$$= i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\,d\theta$$

This is true for any $r > 0$ however small. Hence as $r \to 0$ we get

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$$\int_C \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a)\,d\theta$$

$$= i\,f(a)\int_{\theta=0}^{2\pi} 1.d\theta$$

$$= i f(a)\,[\theta]_0^{2\pi}$$

$$= i\,f(a)\,(2\pi-0)$$

$$\int_C \frac{f(z)}{z-a}\,dz = i\,2\pi\, f(a)$$

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$ //

Cauchy's integral formula.

Generalized Cauchy's integral formula:

Statement: If $f(z)$ is analytic inside and on a simple closed curve C & if 'a' is a point within C then

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Proof: we have Cauchy's integral formula

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}\,dz$$

Applying Leibnitz rule for differentiation under the integral sign we have

$$f'(a) = \frac{1}{2\pi i}\int_C f(z)\,\frac{\partial}{\partial a}\left(\frac{1}{z-a}\right) dz$$

$$= \frac{1}{2\pi i}\int_C f(z) \times \frac{-1}{(z-a)^2} \times -1 \; dz$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\, dz$$

$$f'(a) = \frac{1!}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\, dz \quad - \text{②}$$

again apply Leibnitz rule for ②

$$f''(a) = \frac{1!}{2\pi i}\int_C f(z)\,\frac{\partial}{\partial a}\left(\frac{1}{(z-a)^2}\right) dz$$

$$= \frac{1!}{2\pi i}\int_C f(z) \times \frac{-2}{(z-a)^3} \times -1 \; dz$$

$$= \frac{1!}{2\pi i}\int_C \frac{2\, f(z)}{(z-a)^3}\, dz$$

$$f''(a) = \frac{2!}{2\pi i}\int_C \frac{f(z)}{(z-a)^3}\, dz$$

Continuing like this after differentiating n times we obtain

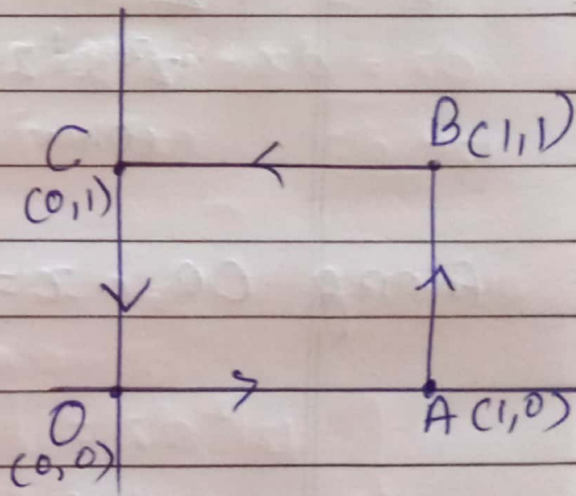
$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz$$

$f^{(n)}(a)$ denotes the n^{th} derivative of $f(z)$ at $z = a$.

Problems

① Verify Cauchy's theorem for the function $f(z) = z^2$ where C is the Square having Vertices (0, 0) (1, 0) (1, 1) (0, 1)

Sol^n C is the Square OABC and we have by Cauchy's theorem



$$\int_C f(z)\,dz = 0$$

∴ we have to S.T

$$\int_{OA} z^2\,dz + \int_{AB} z^2\,dz + \int_{BC} z^2\,dz + \int_{CO} z^2\,dz = 0$$

Along OA, $y=0$, $dy=0$, $z^2 dz = (x+iy)^2\, dx + i\,dy$

$$z^2\,dz = x^2\,dx$$

where $0 \le x \le 1$

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Along AB, $x=1$ $dx=0$, $z^2 dz = (x+iy)^2 dx + idy$

$$z^2 dz = (1+iy)^2 i\,dy$$

$$z^2 dz = (1+i^2y^2+2iy)\,i\,dy$$

$$z^2 dz = (1-y^2+2iy)\,i\,dy$$

where $0 \le y \le 1$

Along BC, $y=1$, $dy=0$ $z^2 dz = (x+iy)^2 dx+i\,dy$

$$z^2 dz = (x+i)^2 dx$$

$$z^2 dz = (x^2+i^2+2xi)\,dx$$

$$z^2 dz = (x^2-1+2xi)\,dx$$

where $1 \le x \le 0$

Along CO, $x=0$, $dx=0$ $z^2 dz = (x+iy)^2 (dx+idy)$

$$z^2 dz = (iy)^2 (i\,dy)$$

$$z^2 dz = -y^2 i\,dy$$

where $1 \le y \le 0$

put all these in ①

① becomes

$$\int_0^1 x^2 dx + i\int_0^1 (1-y^2+2iy)\,dy + \int_1^0 (x^2-1+2xi)\,dx + \int_1^0 -y^2 i\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 + i\left[y - \frac{y^3}{3} + i2\frac{y^2}{2}\right]_0^1 + \left[\frac{x^3}{3} - x + i2\frac{x^2}{2}\right]_1^0 - i\left[\frac{y^3}{3}\right]_1^0$$

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$= \left(\frac{1}{3} - 0\right) + i\left(1 - \frac{1}{3} + i\right) + \left(0 - \left(\frac{1}{3} - 1 + i\right)\right)$

$- i\left(0 - \frac{1}{3}\right)$

$= \cancel{\frac{1}{3}} + \cancel{i} - \cancel{\frac{i}{3}} + i^2 \cancel{- \frac{1}{3}} + 1 \cancel{- i} \cancel{+ \frac{i}{3}}$

$= -i^2 + 1$

$= -1 + 1$

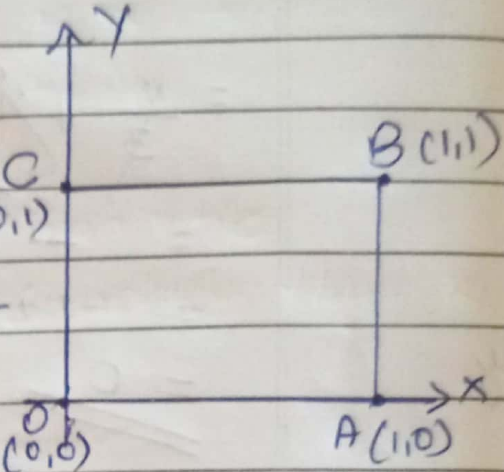
$= 0$

$\int_C f(z)\, dz = 0$

hence Cauchy's theorem is verified

(2) S.T $\int_C |z|^2 dz = i-1$ where C is the square having vertices $(0,0)\,(1,0)\,(1,1)\,(0,1)$. Give reason for Cauchy's theorem not being satisfied.

Solⁿ



C is a square OABC

$$\int_C |z|^2 dz = \int_{OA} |z|^2 dz + \int_{AB} |z|^2 dz + \int_{BC} |z|^2 dz + \int_{CO} |z|^2 dz \quad \text{—} (1)$$

$z = x+iy$

$|z| = \sqrt{x^2+y^2}$

$|z|^2 = (\sqrt{x^2+y^2})^2$

$|z|^2 = x^2+y^2$

$|z|^2 = x^2+y^2$

$dz = dx + i\,dy$

$|z|^2 dz = (x^2+y^2)(dx+i\,dy)$

Along OA, $y=0,\ dy=0$ $\ |z|^2 dz = x^2 dx$, where $0 \le x \le 1$

$$\int_{OA} |z|^2 dz = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \left(\frac{1}{3} - 0\right) = \frac{1}{3}$$

Along AB, $x=1,\ dx=0$ $\ |z|^2 dz = (1+y^2)\,i\,dy$

where $0 \le y \le 1$

$$\int_{AB} |z|^2 dz = \int_0^1 (1+y^2)\,i\,dy = i\int_0^1 (1+y^2)\,dy = i\left[y + \frac{y^3}{3}\right]$$

$$= i\left[1 + \frac{1}{3} - 0\right]$$

$= i\,4/3$

Along BC, $y=1$ $dy=0$ $|z|^2 dz = (x^2+1)\,dx$

where $1 \le x \le 0$

$$\int_{BC} |z|^2 dz = \int_1^0 (x^2+1)\,dx = \left[\frac{x^3}{3}+x\right]_1^0 = \left[0-(1/3+1)\right]$$

$$= -4/3$$

Along CO, $x=0, dx=0$ $|z|^2 dz = y^2\, i\,dy$

where $1 \le y \le 0$

$$\int_{CO} |z|^2 dz = \int_1^0 y^2\, i\,dy = i\left[y^3/3\right]_1^0 = i\left[0-1/3\right]$$

$$= -i/3$$

put all thse in ①

$$\int_C |z|^2 dz = 1/3 + i\,4/3 - 4/3 - i/3$$

$$= (1/3 - 4/3) + i\,(4/3 - 1/3)$$

$$= -3/3 + i\,3/3$$

$$\int_C |z|^2 dz = -1+i \quad \text{or} \quad i-1$$

According to Cauchy's theorem

we have to S.T $\int_C f(z)\,dz = 0$

here $f(z) = |z|^2$

$u + iv = x^2 + y^2$

$u = x^2 + y^2 \quad v = 0$

$u_x = 2x \quad v_x = 0$

$u_y = 2y \quad v_y = 0$

C-R eqn not satisfied

hence $f(z) = |z|^2$ is not analytic

This is the reason for Cauchy's theorem not being satisfied.

(3) Verify Cauchy's theorem for the function $f(z) = ze^{-z}$ over the unit O'e with origin as the centre.

Soln: we have to evaluate $\int_C ze^{-z} dz$ where C is the O'e $|z| = 1$

$z = re^{i\theta}, \quad 0 \le \theta \le 2\pi \quad dz = ie^{i\theta} d\theta$

$z = e^{i\theta}$

$$\int_C ze^{-z} dz = \int_{\theta=0}^{2\pi} e^{i\theta} e^{-e^{i\theta}} ie^{i\theta} d\theta$$

$$= i \int_{\theta=0}^{2\pi} e^{2i\theta} e^{-e^{i\theta}} d\theta$$

put $e^{i\theta} = t \quad \therefore e^{i\theta} i d\theta = dt$ or $d\theta = dt/it$

when $\theta = 0, \; t = e^0 = 1$

$\theta = 2\pi, \; t = e^{2\pi i} = \cos 2\pi + i \sin 2\pi$

$t = 1$

$$\int_C z e^{-z} dz = \int_{t=1}^{1} t^2 e^{-t} \frac{dt}{t}$$

$$= \int_{t=1}^{1} t e^{-t} dt$$

Since the limits are same, the value of the integral is zero.

$\int_C z e^{-z} = 0$ Cauchy's theorem is verified.

Problems on Cauchy's Integral formula

Working procedure:-

1. We need to evaluate integrals of the form $\int_C \frac{f(z)}{z-a} dz$; $\int_C \frac{f(z)}{(z-a)^{n+1}} dz$ over a given closed curve C.
2. Firstly we have to find out whether the point $z=a$ lies inside or outside the given curve C.
3. If $z=a$ is inside C then we use Cauchy's integral formula in the forms $\int_C \frac{f(z)}{z-a} dz = 2\pi i f(a)$, $\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$.
4. If the point $z=a$ is outside C and supposing that $F(z) = \frac{f(z)}{z-a}$ or $\frac{f(z)}{(z-a)^{n+1}}$ is analytic

inside and on the given curve C
we can conclude that $\int_C f(z)\,dz = 0$
by Cauchy's theorem

* In other words if $z=a$ is outside C
the value of integral is zero

problems

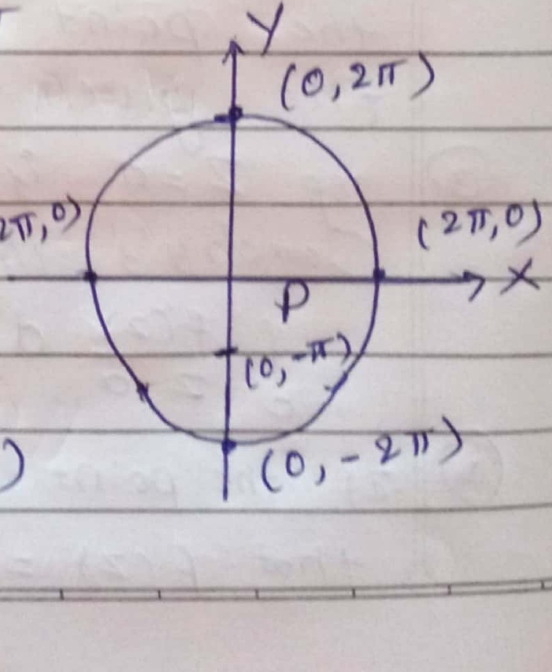
① Evaluate $\int_C \frac{e^z}{z+i\pi}\,dz$ over each of the
following contours C:
ⓐ $|z| = 2\pi$ ⓑ $|z| = \pi/2$ ⓒ $|z-1| = 1$

Sol^n: $\int_C \frac{e^z}{z+i\pi}\,dz$

$\int_C \frac{e^z}{z-(-i\pi)}\,dz$ is of the form $\int_C \frac{f(z)}{z-a}\,dz$

here $f(z) = e^z$ $a = -i\pi$

ⓐ $|z| = 2\pi$ is
a circle with centre
$(0,0)$ and radius 2π
The point $z = a = -i\pi$
is the point $P(0,-\pi)$



lies within the circle $|z| = 2\pi$

we have C.I.F

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

w.K.T $f(z) = e^z$, $a = -i\pi$

$$\int_C \frac{e^z}{z+i\pi}\,dz = 2\pi i f(-i\pi) = 2\pi i e^{-i\pi}$$

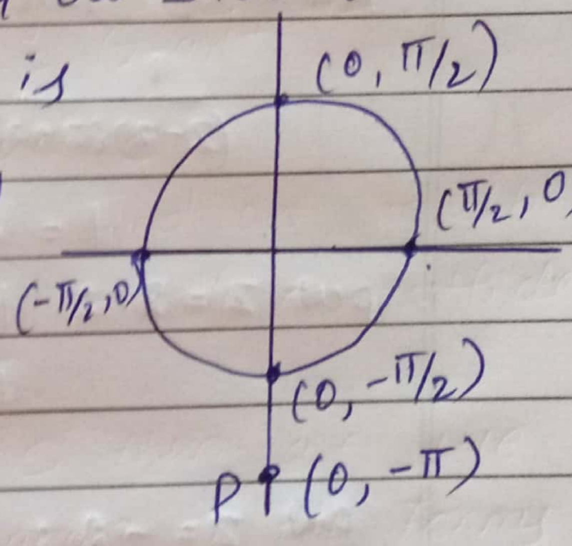
$$= 2\pi i(\underbrace{\cos\pi}_{(-1)} - i\underbrace{\sin\pi}_{0})$$

$$\int_C \frac{e^z}{z+i\pi}\,dz = -2\pi i$$

(b) $|z| = \pi/2$

is a circle with centre origin and radius $\pi/2$

The point $(0, -\pi)$ lies outside the circle $|z| = \pi/2$ and $\frac{e^z}{z+i\pi}$ is analytic inside and on the circle $|z| = \pi/2$



By Cauchy's theorem

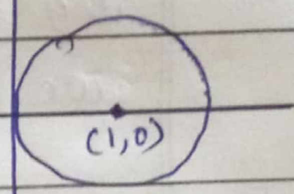
$$\int_C \frac{e^z}{z+i\pi}\,dz = 0$$

(c) $|z-1| = 1$ is a circle with centre at $z = a = 1$ and radius 1. This is a circle with centre $(1, 0)$ & radius 1.

The point $P(0, -\pi)$ lies outside the $\odot^{le}$

$\therefore$ by Cauchy's theorem

$$\int_C \frac{e^z}{z+i\pi}\, dz = 0$$



(2) Evaluate $\int_C \frac{dz}{z^2-4}$ over the following curves C.

(a) $C: |z| = 1$ (b) $C: |z| = 3$ (c) $C: |z+2| = 1$

Sol^n: Consider $\frac{1}{z^2-4} = \frac{1}{z^2-2^2} = \frac{1}{(z-2)(z+2)}$

Resolving into partial fractions

$$\frac{1}{(z-2)(z+2)} = \frac{A}{z-2} + \frac{B}{z+2}$$

$$1 = A(z+2) + B(z-2)$$

put $z = 2$: $1 = 4A + 0$

$1 = 4A \Rightarrow A = 1/4$

put $z = -2$: $1 = 0 + B(-4)$

$1 = -4B$

$B = -1/4$ //

$$\frac{1}{(z-2)(z+2)} = \frac{1}{4(z-2)} - \frac{1}{4(z+2)}$$

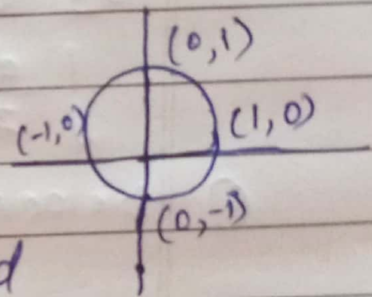
$$\int \frac{1}{(z-2)(z+2)}\,dz = \frac{1}{4}\int \frac{1}{z-2}\,dz - \frac{1}{4}\int \frac{1}{z-(-2)}\,dz$$

—— ①

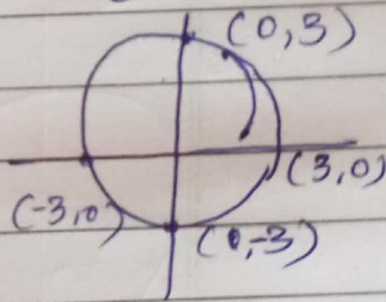
ⓐ $C : |z| = 1$; $z = a = 2$ and $z = a = -2$ lies outside the circle.

$\therefore$ by Cauchy's theorem

$\int_C \frac{dz}{z^2-4} = 0$ where $C : |z| = 1$ //



ⓑ $C : |z| = 3$; $z = a = 2$ and $z = a = -2$ lies inside the circle



W.K.T here $f(z) = 1$

from eqⁿ ①

$$\int_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\int_C \frac{f(z)}{z-2}\,dz = 2\pi i f(2) = 2\pi i(1) = 2\pi i$$

$$\int_C \frac{f(z)}{z+2}\,dz = 2\pi i f(-2) = 2\pi i(1) = 2\pi i$$

put these in ①

$$\int_C \frac{dz}{(z-2)(z+2)} = \frac{1}{4}(2\pi i) - \frac{1}{4}(2\pi i) = 0 //$$

$$\int_C \frac{dz}{z^2-4}\,dz = 0 //$$

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(6) $C : |z+2| = 1$

This is a c^le with centre $(-2, 0)$ and radius 1

The point $(2,0)$ lies outside the c^le hence by Cauchy's theorem

$$\int \frac{dz}{z-2} = 0$$

The point $(-2,0)$ lies inside the c^le by Cauchy integral formula

$$\int \frac{dz}{z+2} = \int \frac{dz}{z-(-2)} = 2\pi i f(-2) = 2\pi i(1) = 2\pi i$$

(1) becomes $\int \frac{dz}{z^2+4} = \frac{1}{4}(0) - \frac{1}{4}(2\pi i)$

$$= -\frac{\pi i}{2}$$

③ Evaluate $\int_C \frac{e^{2z}}{(z+1)(z-2)} dz$ where C is the $\circ^{le}$ $|z| = 3$

Sol^n:

$$\int \frac{e^{2z}}{(z+1)(z-2)} dz$$

Let $\frac{1}{(z+1)(z-2)} = \frac{A}{z+1} + \frac{B}{z-2}$ —— ⊛

$1 = A(z-2) + B(z+1)$ —— ①

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put $z = 2$ in ①

$1 = 0 + B(3)$

$3B = 1 \Rightarrow B = 1/3$

put $z = -1$ in ①

$1 = A(-1-2) + 0$

$1 = -3A$

$A = -1/3$

(*) becomes

$$\int \frac{e^{2z}}{(z+1)(z-2)} dz = -\frac{1}{3}\int \frac{1}{z+1} + \frac{1}{3}\frac{1}{z-2}$$

$$\frac{1}{(z+1)(z-2)} = \frac{1}{3}\frac{1}{z-2} - \frac{1}{3(z+1)}$$

$$\int \frac{e^{2z}}{(z+1)(z-2)} dz = \frac{1}{3}\int \frac{e^{2z}}{z-2} dz - \frac{1}{3}\int \frac{e^{2z}}{\underbrace{z+1}_{z-(-1)}} dz$$

$|z| = 3$ is a $\bigcirc^{le}$ with centre $(0,0)$ & radius 3.

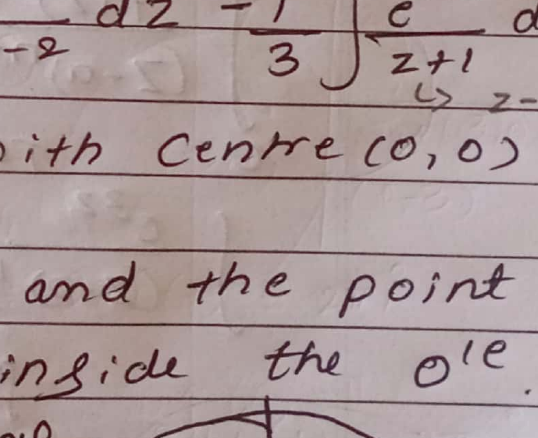
The point $z = a = 2$, and the point $z = a = -1$ both lie inside the $\bigcirc^{le}$.

hence by Cauchy integral formula, w.k.t $f(z) = e^{2z}$

$$\int \frac{e^{2z}}{z-2} dz = 2\pi i f(2)$$

$$= 2\pi i e^{2\times 2}$$

$$= 2\pi i e^4$$



No need

Just refrence

$$\int \frac{e^{2z}}{z+1}\,dz = \int \frac{e^{2z}}{z-(-1)}\,dz = 2\pi i f(-1) = 2\pi i e^{-2} = \frac{2\pi i}{e^2}$$

put in ①

① becomes

$$\int \frac{e^{2z}}{(z+1)(z-2)}\,dz = \frac{2\pi i e^4}{3} - \frac{2\pi i}{3e^2}$$

$$= \frac{2\pi i}{3}\left(e^4 - 1/e^2\right)$$

(ii) Evaluate $\int \frac{e^{3z}}{z^2}\,dz$ over $C: |z| = 1$

Solⁿ: $|z|=1$ is a circle with the centre $(0,0)$ and radius 1. The point $z=0$ lies inside the circle & we have by Cauchy integral formula in the generalized form

$$\int_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Take $f(z) = e^{3z}$, $a=0$, $n=1$

$$\int \frac{e^{3z}}{(z-0)^{1+1}}\,dz = \int \frac{e^{3z}}{z^2}\,dz = \frac{2\pi i}{1!} f'(a)$$

$$f'(z) = 3e^{3z}, \quad z=a=0$$

$$f'(0) = 3e^0 = 3$$

$$\int \frac{e^{3z}}{z^2}\,dz = 2\pi i(3) = 6\pi i$$

(5) Evaluate $\int \frac{z^2+z+1}{(z-2)^3} dz$ over $C : |z| = 3$

Soln: The $|z|=3$ is a circle with centre $(0,0)$ & radius 3. The point $z=2$ lies inside the circle. $\therefore$ by Cauchy's integral formula

$$\int \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$$

Taking $f(z) = z^2+z+1$

$$f'(z) = 2z+1$$
$$f''(z) = 2$$

$$\int \frac{z^2+z+1}{(z-2)^3} dz = \int \frac{z^2+z+1}{(z-2)^{\overset{n+1}{2+1}}} dz = \frac{2\pi i}{2!} f^{(2)}(a) \rightarrow f''(a)$$

$$z = a = 2$$

$$f''(z) = f''(2) = 2$$

$$\int \frac{z^2+z+1}{(z-2)^3} dz = \frac{2\pi i}{2!} f''(2) = \frac{2\pi i}{\cancel{2}} \times \cancel{2} = 2\pi i$$

(6) Evaluate $\int \frac{e^{\pi z}}{(2z-i)^3} dz$ where C is circle $|z| = 1$

Soln: $|z|=1$ is a circle with centre $(0,0)$ & radius is 1.

$$\int \frac{e^{\pi z}}{(2z-i)^3} dz = \int \frac{e^{\pi z}}{\left(2\left(z-\frac{i}{2}\right)\right)^{2+1}} \rightarrow \text{express in terms of } (z-a)^{n+1}$$

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The point $z = a = i/2$ i.e $(0, 1/2)$ lies inside the o'e

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by Cauchy integral Generalized formula

$$\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \int \frac{e^{\pi z}}{2^3 (z - i/2)^{2+1}} dz = \frac{1}{8} \times \frac{2\pi i}{2!} f''(a)$$

$$f(z) = e^{\pi z}$$

$$f'(z) = \pi e^{\pi z}$$

$$f''(z) = \pi^2 e^{\pi z}$$

$$z = a = -i/2$$

$$f''(-i/2) = \pi^2 e^{-i\pi/2}$$

$$\Rightarrow \frac{1}{8}\int \frac{e^{\pi z}}{(z - i/2)^3} = \frac{1}{8} \times \frac{\cancel{2}\pi i}{\cancel{2}!} f''(a)$$

$$= \frac{1}{8} \pi i f''(-i/2)$$

$$= \frac{1}{8} \pi i \cdot \pi^2 e^{-i\pi/2}$$

$$= \frac{1}{8} i \pi^3 e^{-i\pi/2}$$

$$= \frac{1}{8} i \pi^3 (\cos \pi/2 - i \sin \pi/2)$$

$$= \frac{1}{8} i^3 \pi^3 (0 - i(1))$$

$$= \frac{1}{8} \times -i^4 \pi^3$$

$$= -\pi^3/8$$

(7) Evaluate $\int_C \frac{e^{2z}}{(z+1)^2(z-2)}\,dz$ where $C: |z|=3$

— ⊛

Sol^n: we shall first resolve $\frac{1}{(z+1)^2(z-2)}$ into partial fractions.

$$\frac{1}{(z+1)^2(z-2)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z-2}$$

$$1 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$$

put $z=-1$

$$1 = 0 + B(-3) + 0$$

$$-3B = 1 \Rightarrow B = -1/3$$

put $z = 2$

$$1 = A(0) + 0 + C(3)^2$$

$$1 = 9C \Rightarrow C = 1/9$$

put $z = 0$

$$1 = A(1)(-2) + B(-2) + C(1)$$

$$1 = -2A + 2/3 + 1/9$$

$$\cancel{2A = 2/3 + 1/9 - 1}$$

$3 \,|\, 9 \cdot 6$
$\quad 3 \;\; 2$

$$2A = \frac{\frac{1}{3}\times 18 + \frac{1}{9}\times 18 - 18}{18}$$

$$= \frac{3+2-18}{18}$$

$$2A = \frac{-12}{18} \Rightarrow A = \frac{-12}{18\times 2} \Rightarrow A = -$$

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$$\gamma = 2A + 2/3 + 1/9$$

$$2A = 2/3 + 1/9 - 1$$
$$= \frac{6+1-9}{9}$$
$$2A = -2/9$$
$$A = -1/9$$

$$\frac{1}{(z+1)^2(z-2)} = \frac{-1}{9} \times \frac{1}{z+1} - \frac{1}{3} \times \frac{1}{(z+1)^2} + \frac{1}{9} \times \frac{1}{z-2}$$

$\times^{ly}$ e^{2z} & Integrate w.r.to z over c

$$\int \frac{e^{2z}}{(z+1)^2(z-2)} dz = \frac{-1}{9} \int \frac{e^{2z}}{z+1} dz - \frac{1}{3} \int \frac{e^{2z}}{(z+1)^2} dz + \frac{1}{9} \int \frac{e^{2z}}{z-2} dz$$

$$I = I_1 + I_2 + I_3 \quad - (*)$$

$$I_1 = \frac{-1}{9} \int \frac{e^{2z}}{z+1} dz = \frac{-1}{9} \int \frac{e^{2z}}{z-(-1)} dz$$

The point $z = a = -1$ i.e $(-1, 0)$ lies inside the circle $\therefore \int_c \frac{f(z)}{z-a} dz = 2\pi i f(a)$

w.k.T $f(z) = e^{2z}$, $f(\cdot$

$$f(-1) = e^{-2}$$

$$\frac{-1}{9} \int \frac{e^{2z}}{z+1} = 2\pi i f(-1) \times \frac{-1}{9}$$
$$= 2\pi i e^{-2} \times \frac{-1}{9}$$

$$= -\frac{2\pi i e^{-2}}{9}$$

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$$I_2 = -\frac{1}{3}\int \frac{e^{2z}}{(z+1)^2}\,dz$$

$$\Rightarrow \int \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i f^{(n)}(a)}{n!}$$

$$-\frac{1}{3}\int \frac{e^{2z}}{(z-(-1))^{1+1}}\,dz = -\frac{1}{3}\times\frac{2\pi i f'(a)}{1!}$$

$$f(z) = e^{2z}$$

$$f'(z) = 2e^{2z}$$

$$z = a = -1$$

$$f'(-1) = 2e^{-2}$$

$$-\frac{1}{3}\int \frac{e^{2z}}{(z+1)^2}\,dz = -\frac{1}{3}\times 2\pi i \times 2e^{-2}$$

$$= \frac{-4\pi i e^{-2}}{3}$$

$$I_3 = \frac{1}{9}\int \frac{e^{2z}}{z-2}\,dz$$

$z = a = 2$ lies inside the O^{le}

$$\int \frac{f(z)\,dz}{z-a} = 2\pi i f(a)$$

$$= \frac{1}{9}\int \frac{e^{2z}}{z-2}\,dz = \frac{1}{9}\times 2\pi i f(2)$$

w.k.t $f(z) = e^{2z}$

$$f(2) = e^{2\times 2} = e^4$$

$$I_3 = \frac{1}{9}\int \frac{e^{2z}}{z-2}\,dz = \frac{1}{9}\times 2\pi i \times e^4$$

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Now (*) $\Rightarrow$

$$I = \frac{-2\pi i e^{-2}}{9} - \frac{4\pi i e^{-2}}{3} + \frac{2\pi i}{9} e^{4}$$

$$I = \frac{2\pi i}{9}\left(\frac{-1}{9e^{2}} - \frac{2}{3e^{2}} + \frac{e^{4}}{9}\right)$$